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Principles of Probabilistic Regional Mineral Resource Estimation

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Abstract: Five principal sources of uncertainty in quantitative mineral resource estimation are listed and illustrated by means of a simple example (mosaic model) and a case history study for large copper deposits in the Abitibi area of the Canadian Shield. Abitibi copper potential originally was estimated on the basis of 1968 estimates of production and reserves totalling 3.12 Mt Cu. This prognostication now could be evaluated on the basis of 2008 copper production and reserves totalling 9.50 Mt Cu. An earlier hindsight study performed on the basis of 1977 data (totalling 5.23 Mt Cu) showed seven new discoveries occurring either in the immediate vicinities of known deposits or on broad regional copper anomalies predicted from the 1968 inputs. By 1977, the global geographic distribution pattern of large copper deposits in the Abitibi area had stabilized. During the next 30 years, new copper was essentially found close to existing deposits, much of it deeper down in the Earth's crust. In this paper, uncertainties associated with copper ore tonnage are analyzed by comparison of 2008 data with 1968 data using (a) log-log plots of size versus rank, and (b) lognormal QQ-plots. Straight lines fitted by least squares on these plots show that 1968 slopes provide good estimates of 2008 slopes but 1968 intercepts are much less than 2008 intercepts. In each linear log-weight versus log-rank plot, the slope is related to fractal dimension of a Pareto frequency distribution, and in a lognormal QQ-plot it is determined by logarithmic variance. The difference between 2008 and 1968 intercepts represents the increase in copper ore production and reserves from 1968 to 2008. The Pareto model fits actual copper and massive sulphides increase over the past 40 years better than the lognormal frequency distribution model for 10 km×10 km cells on favorable environments in the Abitibi area.

Key words: mineral resources; quantitative estimation; uncertainty; mosaic model; case history study; Abitibi area; copper deposit; Pareto model; lognormal model.

INTRODUCTION

Geoscientists are using both facts and concepts to determine the probable occurrence of mineral deposits. Examples of basic facts are age and lithological data on rock units, chemical determinations and geophysical measurements. There is a gradual transition from basic facts to conceptual projections of structures and composition of rock formations. For this paper, standard conceptual projections normally made geoscientists, e.g. extrapolations to bedrock partially overlain by glacial debris, will be taken as facts. Originally, ore was discovered by prospectors but later economic geologists could narrow their search by using genetic models. Bateman (1919) argued convincingly that the old saying "where ore is, there it is" was to be replaced by answering the question "why ore is where it is". He promoted "intelligently directed search for ore or oil". Today much ore is being found by means of advanced geophysical prospecting techniques. Moreover, as Zhao et al. (2008) pointed out, increasingly new deposits are being discovered at greater depths with the aid of 2D and 3D specialized technologies on the one side (see e.g. de Kemp, 2006) and non-linear modeling on the other (Cheng, 2007, 2008). Both geomathematics and conceptual thinking are needed to extrapolate data from the surface downward into hidden rock formations at greater depths. Such projections remain subject to significant uncertainty that has to be quantified in order to allow valid decision-making.

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The problem considered in this paper is how the mineral potential of a region can be assessed systematically by statistical extrapolation from known facts. Because of the complexity of the geological framework, many authors have employed a variety of more subjective methods for mineral potential estimation, often with good results, but these others, "knowledge-driven" methods (cf. Bonham-Carter, 1994; Bardossy and Fodor, 2004; Carranza, 2008; Singer and Menzie, 2010) are outside the scope of this paper. In the next section, some basic principles of quantitative treatment of the geological framework of a region will be considered first and demonstrated on the basis of a simple, 2D mosaic model. Later, in a case history study, a geomathematical prognosis made from 1968 data for copper potential of the Abitibi area on the Canadian Shield will be reviewed and compared with amounts of copper and copper ore discovered in this area during the past 40 years.

The future of fully automated regional mineral resource estimation is promising because, increasingly, sophisticated geophysical remote sensing techniques are becoming available, while rapid progress is being made in the field of three-dimensional geological mapping. It should be kept in mind, however, that the geological framework generally is highly heterogeneous. In addition to continuous spatial variability observed for geophysical fields, there are numerous discontinuities in the upper Earth crust, e. g. at contacts between different rock units and where there are faults. In general, advanced pre-processing techniques are required to produce realistic 3D images providing the inputs for mineral potential estimation.

MINERAL RESOURCE ESTIMATION

For the purpose of this discussion, it is useful to make a distinction between mineral exploration and mineral resource estimation. The objective of mineral exploration is to delineate high-potential target areas. This can be achieved by ranking cells or pixels by means of a probability index for relative prospectivity. In mineral resource estimation, the primary objective is to predict number of deposits and their sizes for larger regions. Any probability index has to be converted into a probability that is unbiased. Early on, mineral resource estimation problems were considered by relatively few authors including Allais (1957) who used the Poisson model for random spatial distribution of large mineral deposits of any type, Griffiths (1966) advocating use of "unit regional value" lumping different types of metal and hydrocarbon deposits together, and Harris (1965) who quantified geological maps for cells relating "total dollar value" based on all metals to bedrock variables by means of multivariate statistical analysis. A characteristic feature of these early statistical publications was that natural resources of different types were analyzed simultaneously. Such lumping can be advantageous if statistical models have the property of additivity (e. g. a mixture of two spatial Poisson process models is another Poisson process model) but often it is better to incorporate different genetic models into the mineral resource estimation. My own approach to mineral resource estimation was commodity-based (Agterberg, 1971, 1974). It can be summarized as follows:

Various sources of uncertainty have to be considered in mineral resource estimation, and to some extent in exploration. These different types of uncertainty were considered separately and combined with one another when copper and zinc mineral potential maps were constructed for the Abitibi area on the Canadian Shield in the early 1970s (Agterberg et al., 1972) to be reviewed in more detail for copper later in this paper. Today, of course, better answers could be obtained than in the early 1970s, because of both theoretical and computational advances. However, the basic problems to be solved remain the same. The five principal sources of uncertainty are:

(1) The first major source of uncertainty is provided by the probability of occurrence itself. Any point in a study area on a map has probabilities associated with it that a small unit area sur-

rounding it contains mineral deposits of different types (If depth can be considered as a third dimension, unit volumes can be used in addition to unit areas).

(2) The estimated probabilities have variances. Suppose that we are concerned with a single deposit type or commodity in 2D. For small unit areas, the probabilities of occurrence then are very small. For example, suppose that the 10% largest probabilities are approximately 0.01. This does not only mean that a unit area with probability 0.01 would contain a deposit but also that the variance of this probability is 0.01. This intrinsic variance normally exceeds the estimation variance of the probability itself.

(3) Intensity of exploration is a third source of uncertainty. This is a largely unknown variable that is difficult to quantify. Fortunately, uncertainty associated with variable intensity of exploration is much less than the uncertainty intrinsic in the probability itself. However, it should be kept in mind that, from an economic point of view, intensity of exploration can be regarded as the most important variable because it determines the number of undiscovered ore deposits.

(4) A second major source of uncertainty in mineral resource estimation is size distribution of the deposits for which the probabilities of occurrence are being estimated. In general, size of mineral deposits as a random variable covers several orders of magnitude with the largest deposits being exceedingly rare but of utmost economic importance. It should be kept in mind that it is possible that deposit size is positively correlated with probability of occurrence.

(5) Metal grades including cut-off grades are to be considered as well although these can often be incorporated in the definition of deposit type. In general, economic data on past production, various types of reserves and grades are of highest quality for the largest deposits with amount of information diminishing and tending to become unavailable for smaller and lower-grade deposits. Two factors to be considered are that mineral deposits for the same metal may occur in different geological settings and that usually more than a single metal is mined from the same deposit suggesting that total amount of ore also is useful as a variable for estimating probabilities of occurrence together with size frequency distribution modeling.

In order to further illustrate uncertainties (1) and (2), let us take a typical weights-of-evidence (cf. Bonham-Carter, 1994) result for example. The output map with posterior probabilities in weights-of-evidence usually is accompanied by a t-value map. Suppose that the t-value associated with a posterior probability of 0.01 is equal to 4. This would mean that the estimation variance of the probability of 0.01 amounts to (the square of $0.04 = 0.0016$), and this is less than 0.01 representing the intrinsic variance associated with the probability itself.

Uncertainties (1) and (2) can be combined with one another by adding the variances. In the preceding example the combined variance is 0.0116. Suppose now, in the preceding situation, that the probability of a (larger) unit cell is 0.1, it would imply that the intrinsic variance is 0.1, with estimation variance of 0.16, and combined variance of 0.26. It illustrates that for larger unit areas and for larger posterior probabilities, relative uncertainty associated with estimation increases significantly. It is noted that probabilities for groups of adjoining pixels can be added. The resulting sums can be interpreted as probabilities if they are less than 1 but are expected values if they are greater than 1.

Elementary Statistics of the Mosaic Model

A small-scale geological map of bedrock in a region is a mosaic on which mineral deposits are projected as points. A simple example of how one can proceed when information of this type is available is as follows: Suppose a study area contains 1 million pixels of which 20 percent are underlain by "favorable" environment A. There are 10 pixels with mineral deposits in this study area of which 8 are on A. The other 2 are on "unfavorable" $A^{\sim}$ where the "$\sim$" symbol denotes "not". Therefore, the probability that any pixel contains a deposit is $P(D) = 0.000\,01$. The probability that a pixel on A has a deposit can be written as $P(D|A) = 8/200\,000 = 0.000\,04$; likewise, $P(D|A^{\sim}) = 2/800\,000 = 2.5\times10^{-6}$. If a probability of occurrence map is constructed on the basis of this information, it contains 200 000 pixels with probability 0.000 04, and 800 000 with probability 2.5×10^{-6}.

The second type of uncertainty is related to precision of the statistics. When weights-of-evidence modeling is applied, the positive weight for the preceding example is 0.982 representing the natural log of the ratio $P(A|D)/P(A|D^{\sim}) = 0.75/0.281 = 2.669$, and the negative weight is -1.056 representing the natural log of the ratio $P(A^{\sim}|D)/P(A^{\sim}|D^{\sim}) = 0.25/0.719$. Thus, the contrast for this example is 2.04 with approximate standard deviation equal to 1.18. The corresponding t-value of 1.73 is barely significant at the 95% level if a one-sided test is used under the normality assumption. It is interesting to apply other resource estimation techniques to this simple mosaic model as well.

For example, we can fit the linear model $Y = a + b\cdot x$, where Y is a random variable assuming the value of 1 at pixels on "A" where $x=1$, and 0 where $x=0$. Using the method of least squares, this gives $a = 2.5\times10^{-6}$ and $b = 37.5\times10^{-6}$. Obviously, this linear regression model exactly reproduces the two probabilities estimated in the first paragraph of this section. The linear equation can also be used in logistic regression with Y representing the logit of occurrence instead of the probability itself. Application of the LOGDIA program (Agterberg, 1989a) then yields $a = -12.90$ and $b = 2.773$, with variances of 0.50 and 0.624, respectively, and covariance of -0.50. Conversion of logits into probabilities again reproduces $P(D|A) = 0.000\,04$ and $P(D|A^{\sim}) = 2.5\times10^{-6}$. The preceding four methods (probability calculus, weights-of-evidence, linear least squares, and logistic regression) all produce the same estimates of the probabilities (uncertainty type 1). However, they produce slightly different answers for the variances of these probabilities (uncertainty type 2).

Some remarks on other applications pertaining to the mosaic model are as follows. This model was used by Bernknopf et al. (2007) for different rock units with probabilities of occurrence for mineral deposits of different types. Probabilities and expected values were modified according to relative amount of exposure of each rock unit by these authors. In the context of weights-of-evidence modeling, Carranza (2009) asked the question of what would be the optimum pixel size. For the mosaic model, the answer to this question is simply that pixels should be sufficiently small to allow precise estimates of relative areas of rock units on the map. Further size decrease does not affect estimation results when mineral deposits are modeled as points, because of the dichotomous nature of every rock unit represented by a mosaic model.

In Agterberg et al. (1972), stepwise multiple regression analysis was used to estimate a probability index for occurrence of large copper deposits shown schematically in Fig. 1 for most of the study area. The input for explanatory variables primarily consisted of information on rock types systematically quantified for cells measuring 10 km on a side. Geophysical field data at cell centers were used for gravity (Bouguer) and regional aeromagnetic anomaly. Products of variables provided better results than scores for individual variables. The dependent variable used in this multivariate linear model was logarithmically transformed total amount of copper in one or more copper deposits per input cell (also see later for schematic representation in Fig. 2A). Neither of the two geophysical input variables made a significant contribution to the magnitudes of the probabilities that were being estimated. However, in a separate computer experiment using binary (presence-absence) input data only, the variable most strongly correlated with occurrence of large copper deposits was a combination of presence of felsic volcanics at the surface of bedrock and higher than average Bouguer anomaly. This result can be interpreted in terms of a mineral deposit model, because nearly all large

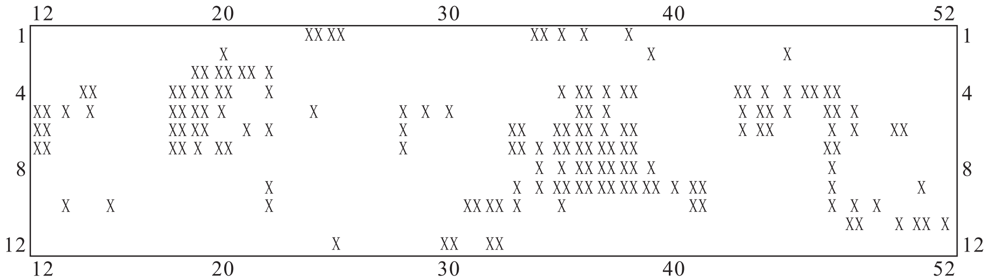

Fig. 1　Pattern of probability index for 10 km×10 km cells with occurrence of large copper deposits in the Abitibi area of the Canadian Shield using 1968 mineral deposit data；single X denotes probability index greater than 4 (and <8)；XX for cells with probability index >8. Probability index values and numbering of cells after Agterberg (1971，Appendix 3)

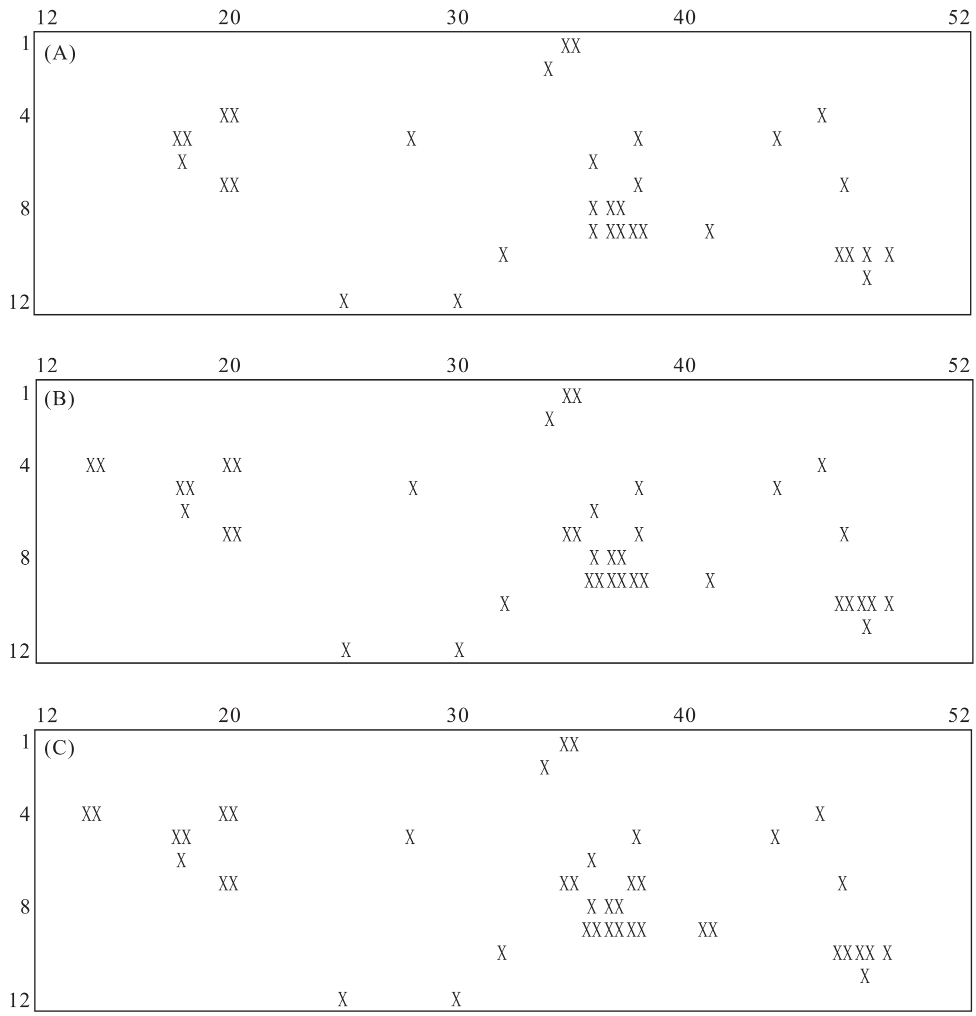

Fig. 2　Pattern comparison for 10 km×10 km cells with one or more large copper deposits in (A) 1968，(B) 1977 and (C) 2008. Original 1968 figures for production and reserves reported in short tons (st) were converted into tons (t). Single X denotes one or more deposits with copper production ＋ reserves (Cu) between 1 000 short tons (st) of but less than 50 000 tons (t)；XX for cells with 50 000 t<Cu (1 t=0. 907 184×1 st). Numbering of cells as in Fig. 1

copper deposits in Abitibi are of the volcanogenic massive sulphide (VMS) type formed near volcanic centers in association with felsic volcanics，while a relatively high Bouguer anomaly on the Canadian Shield indicates relatively large amounts of mafic volcanic rocks with above average specific gravity at greater depths. The probability index map schematically shown in Fig. 1 was converted into prob-

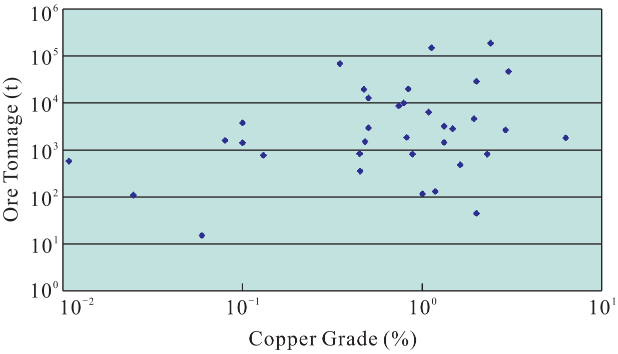

Fig. 3 Log-Log Ore Tonnage-Copper Grade plot (2008 data). The three points on the left may be outliers. When these 3 points are deleted, the correlation coefficient ($r=0.079$) is nearly zero suggesting lack of functional relationship between grade and ore tonnage

abilities of occurrence, which were added for larger 40 km×40 km unit areas to produce a prognostic contour map for expected numbers of known and unknown copper deposits. The difference between predicted and known occurrences for a relatively well explored "control area" was shown on a separate map displaying three broad subareas with relatively high copper potential (Agterberg, 1971, Fig. 3). Later, Assad and Favini (1980) performed statistical mineral exploration for the eastern part of the Abitibi study region using localized geophysical (aeromagnetic, gravity and terrain elevation) anomalies only. This prompted me (Agterberg, 1989b) to use the probabilities previously obtained in the Abitibi case study as priors in weights-of-evidence modeling incorporating Favini and Assad's map layers based on the more local geophysical anomalies in order to obtain more local posterior probabilities of occurrence.

ABITIBI COPPER POTENTIAL PREDICTION HISTORICAL CASE STUDY

The Abitibi area copper potential map constructed in 1971 was based on 1968 statistics for production and reserves (Agterberg et al., 1972). During the 1970s a considerable amount of exploration for additional massive sulphides was undertaken in this region. Agterberg and David (1979) evaluated the prognostic copper potential contours on the basis of the locations and sizes of seven discoveries made between 1966 and 1977 (Millenbach, Louvem, Conigo, Iso-Copperfield, New Insco, Corbet and Montcalm Ni-Cu deposits). The first three of these deposits had been discovered when the original statistical analysis was performed but published figures on production and reserves were not yet available for them. All seven new discoveries occurred either within the vicinity of one or more of the original set of 41 deposits, or on the three relatively high copper potential subareas mentioned before (also see Wellmer, 1983). Together the 41 deposits contained 3.12 Mt of copper at the end of 1968. In 1977, the set of (41+7=) 48 deposits contained 5.23 Mt Cu. This increase was largely due to increased production and reserve estimates for the Kidd Creek mine (near Timmins, Ontario). The overall change in geographic distribution of large copper deposits from 1968 to 1977 can be seen by comparing Fig. 2B with Fig. 2A.

More recently, Lydon (2007) has published a comprehensive overview of the economic and geological contexts of Canada's major mineral deposit types accompanied by a DVD with production and estimated reserves of Canadian mineral deposits including large copper deposits in the Abitibi area. Between 1977 and 2007, there were five major new discoveries (Ansil, Bouchard-Hebert, Bousquet-Laronde, Amos and Louvicourt deposits), all within the vicinities of the 48 deposits known to exist in 1977. Revising our original 1968 data set (Agterberg et al., 1972, Appendix 1) using Lydon's dataset, and including one 2008 estimate for the newly discovered Upper Beaver ore zone near Kirkland Lake, Ontario (cf. www.queenston.ca/news/pdf/080922.pdf), yields a combined set of 66 copper deposits containing 9.50 Mt Cu, about three times as much as the 41 copper deposits in 1968. The 41 copper deposits in the 1968 dataset occur in 27 "copper cells" measuring 10 km on a side and belonging to the original set of 644 cells for which copper potential was estimated in

1971. Fig. 2 shows most copper cells, with (A) locations of 1968 copper cells, (B) locations of 1977 copper cells including 2 new cells with new discoveries, and (C) locations of copper cells in 2008 each containing one or more copper deposits in the combined data set. The 1968 data set has 3 deposits not in Lydon's data base but plotted in Fig. 2, whereas the Lydon data base contains 3 Ni-Cu deposits with 0.1% copper grade not in Agterberg et al. (1972, Appendix 1) and not plotted in Fig. 2. On average, a copper cell shown in Fig. 2C contains (63/35) = 1.80 large copper deposits but because of localized strong spatial clustering, the frequency distribution of number of deposits per copper cell is highly positively skewed with one cell (37,8) in the Noranda mining district containing as many as 11 large copper deposits. Comparison with Fig. 2B shows that nearly all differences between the patterns in Figs. 1A and 1C date from before 1977. As mentioned before, new discoveries during the 1970s either were close to known 1968 deposits or within the areas with relatively high copper potential outlined in Fig. 1. By 1977, geographical distribution of large copper deposits in the Abitibi area had stabilized and further increases in production and reserves (from 5.23 to 9.50 Mt) were for copper within known deposits and for new discoveries close to known deposits. Average grade of total production and reserves is about 1.6% copper in the original 1968 data set with 41 copper deposits as well as in the 2008 data set with 66 copper deposits. Fig. 3 is a log-log plot of copper grade versus amount of ore for the corresponding 35 copper cells that also will be analyzed later in this paper.

Predictions made in Agterberg et al. (1972), such as the one for a "test area" in the surroundings of Timmins, Ontario, were based on the assumption that the frequency distribution of amount of copper per control cell could be used for this purpose. As mentioned in the introduction, a relatively recent development is that increasingly it is realized that ore deposits, like earthquakes and several other types of natural phenomena, have fractal characteristics and resulted from non-linear processes. Mandelbrot (1983, p. 263) posed a challenge to geoscientists by stating that oil and other natural resources have Pareto distributions and "this finding disagrees with the dominant opinion, that the quantities in question are lognormally distributed. The difference is extremely significant, the reserves being much higher under the hyperbolic than under the lognormal law." This topic will now be investigated in more detail for copper in the Abitibi area.

Comparison of Weight Frequency Distributions for Copper Metal and Ore Contained in (10 km × 10 km) Cells in the Abitibi Area

Size frequency distribution studies usually are carried out on populations of mineral deposits of the same type. In this study, it is applied to total amount of copper in the one or more copper deposits per 10 km×10 km cell. This procedure has advantages as well as drawbacks. An advantage is that the effect of strong localized clustering of deposits is reduced, and total number of observations (27 in 1971 versus 35 in the combined data set) is stabilized. A disadvantage is that copper deposits of different types are being combined with one other although frequency distributions for different types of deposits can be different, especially if two or more metals are used.

Nearly all (86% of 66) large copper deposits in the combined data set are VMS. There are relatively few (6) magmatic Ni-Cu deposits (and 5 of these are small), plus three porphyry-type copper deposits. Preliminary statistical analysis was performed on various subsets such as using copper deposits instead of copper cells, VMS deposits only, Lydon's statistics only, but these exercises produced results similar to those to be presented here. However, explicit consideration of average copper grade (= amount of copper/amount of ore) generates somewhat different results. For this reason, the following analysis is for two variables per copper cell: (1) total weight (amount) of copper, and (2) total weight of ore. A comparison will be made between the 2008 and 1968 data.

Figure 4 shows log-log plots of copper and ore weight versus rank. A Pareto distribution plots approximately as a straight line on this type of plot as previously shown for gold tonnages in lode gold deposits in the Superior Province of the Canadian Shield (Agterberg, 1995). In each plot of Fig. 4, a straight line was fitted by least squares to most data points excluding the smallest copper or copper ore cells for which information is probably incomplete. Also, it can be expected that the Pareto distribution does not provide a good fit for low weight cells because it has the property that frequency of occurrence continues to increase with decreasing weight. Figure 5 shows the corresponding four lognormal QQ-plots. In Figs. 5a and 5c for copper weight, the patterns are not linear but in Figs. 5b and 5d they are, and straight lines were fitted by least squares using all data points. Degree of fit is good in these two diagrams as illustrated by the 95% confidence interval in Fig. 5d.

It may be concluded that six of the eight plots (Figs. 4 and 5) show straight line patterns. The patterns in Figs. 6a and 6b are not approximately linear, probably because in several deposits copper is not the main metal of economic interest but mineable as a by-product. For these deposits, total weight of ore fits in with the population of all copper deposits but total weight of copper does not because of the lower copper grades. It seems that both the Pareto and the lognormal are good candidates for modeling total copper and ore weight frequency distributions. The high-value tail of a Pareto frequency distribution is thicker than that of the lognormal. As discussed in more detail elsewhere (e. g. Agterberg, 2007), the Pareto and lognormal each can be considered as the end product of a multiplicative cascade model. They are generated by a de Wijs cascade and a Turcotte cascade, respectively. Other cascades (cf. Lovejoy and Schertzer, 2007) can result in frequency distributions that resemble a lognormal except in their high-value Pareto-type tails. It may not be possible to determine whether a high-value tail is lognormal or Pareto-type if a frequency distribution has strong positive skewness like the distributions of Figures 5 and 6, because then there are too few very large values for

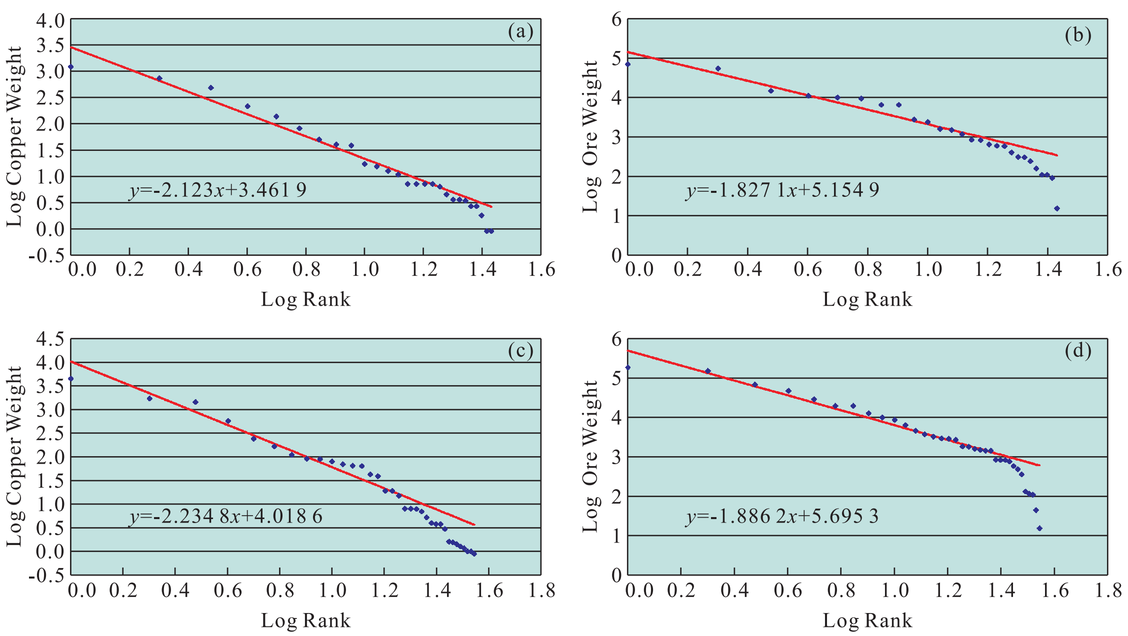

Fig. 4 Log-Log Weight-Rank plots for 1968 and 2008 data with straight lines fitted by least squares. (a) 1968 Copper Weight; (b) 1968 Ore Weight; (c) 2008 Copper Weight; (d) 2008 Ore Weight. Base of logarithm= 10; Weight measured in (metric) kilotons. Straight line approximates Pareto frequency distribution with fractal dimension estimated by inverse of slope. For 1968 data, first 18 of 27 data points were used to fit straight lines. For 2008 data, first 27 of 35 data points were used to fit straight lines

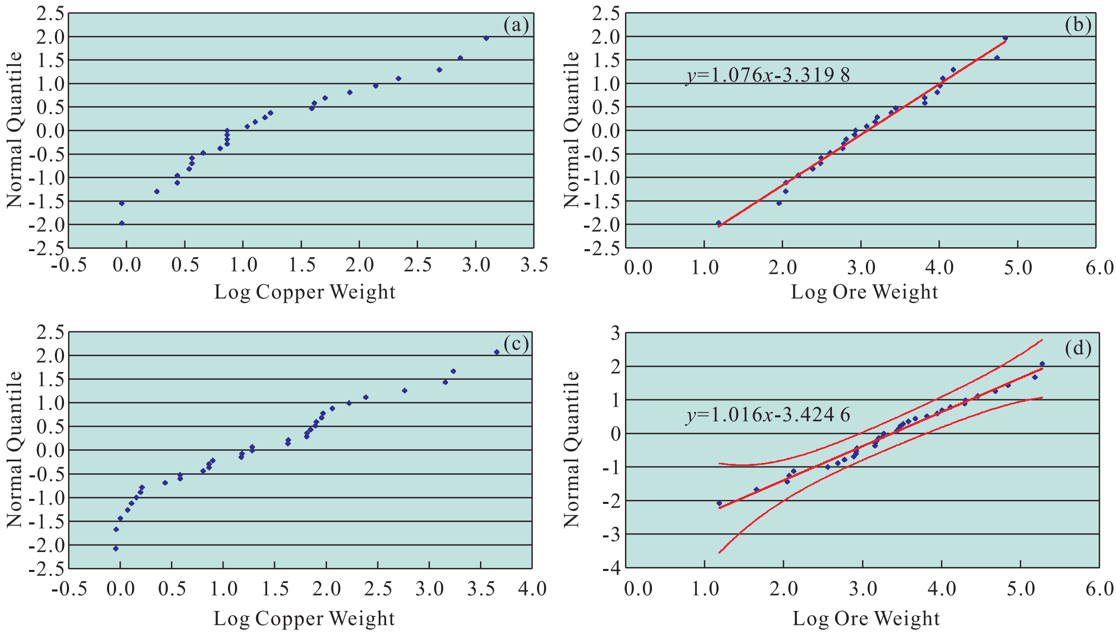

Fig. 5　Lognormal QQ-plots of copper and ore weights for 1968 and 2008 data with straight lines for ore weights fitted by least squares. (a) 1968 Copper Weight; (b) 1968 Ore Weight; (c) 2008 Copper Weight; (d) 2008 Ore Weight. Base of logarithm = 10; Weight measured in (metric) kilotons. Straight line approximates lognormal frequency distribution with logarithmic standard deviation estimated by inverse of slope. Curves in Fig. 5d represent 95% confidence belt for points deviating randomly from straight line. All data points were used to fit straight lines

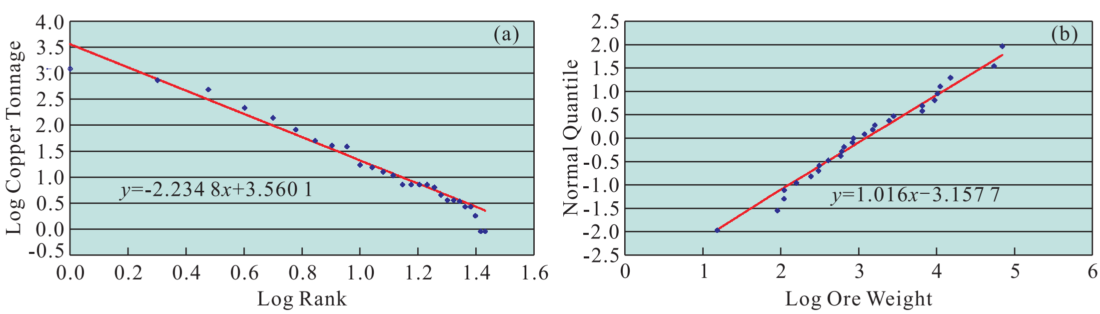

Fig. 6　Best-fitting straight lines for 1968 data with slopes set equal to slopes of straight lines fitted to 2008 data. (a) Log-Log Copper Weight; points same as in Fig. 4a; (b) Lognormal QQ-plot of Ore Weight; points same as in Fig. 5b. Comparison with Figs. 4c and 5d shows 1968 to 2008 intercept increases

application of standard goodness-of-fit tests (also see Agterberg, 1995). However, the few largest values contribute much or most of total weight for all deposits in the data set. This dilemma can be resolved by using the following new method of comparing the Pareto and lognormal frequency distributions with one another.

Comparison of coefficients of the straight lines fitted in Figs. 4 and 5 shows that 1968 and 2008 data have approximately the same slope but 2008 intercepts are markedly greater than 1968 intercepts. In Fig. 4 the straight lines were based on 18 or 27 data points for 1968 and 2008, respectively. All data points were used for the lognormals of Fig. 5. Because slope differences are small, it can be assumed that 1968 slopes are unbiased estimates of 2008 slopes. This is illustrated in Fig. 6 for 1968 copper and ore weight data where the best-fitting lines were forced to have the 2008 slopes. The intercept of the Pareto distribution for copper weight in Fig. 6a is less than its intercept in Fig. 4c. This difference can be written as $\Delta_P = 0.458\,5$.

The equations of the straight lines in Figs. 4 to 6 are of the form $y = bx + a$ indicating that the dependent variable (y) was regressed on the explanatory variable (x). All uncertainty is assumed to be associated with y that is plotted in the vertical direction. These equations can be rewritten as $x = b'y + a'$; for example, $x = 0.984\,3y + 3.373\,7$ for Fig. 5d and $x = 0.984\,3y + 3.108\,0$ for Fig. 6b.

The intercept (a') of the lognormal distribution in Fig. 6b is only $\Delta_L = 0.262\,7$ less than the intercept in Fig. 5d. Each intercept difference Δ corresponds to a factor of 10^{Δ} for increase in average weight per cell between 1968 and 2008. These factors are 2.875 for copper and 1.831 for copper ore, respectively. Incorporating a 6.1% correction related to the increase in total number of cells with known deposits due to new discoveries (cf. Fig. 2), the factors of increase in total weight become 3.049 for the copper-weight Pareto model, and 1.943 for the ore-weight lognormal model. Observed factors of increase are 3.026 for total copper weight and 3.030 for total ore weight, respectively. Consequently, the copper-weight Pareto agrees better with observed increase in total copper weight than the ore-weight lognormal, which significantly underestimates observed overall change in total ore weight. As an additional test, it was determined from the straight lines in Figs. 4b and 4d, that $\Delta_P = 0.488\,5$ for total ore weight, resulting in an increase factor of 3.266, slightly overestimating the observed value of 3.030.

CONCLUDING REMARKS

Five principal sources of uncertainty in quantitative mineral resource estimation were listed at the beginning of this paper. The first two uncertainties are related to probability of occurrence of a mineral deposit in a small unit area within a study area, and these were illustrated by means of a simple example (mosaic model). The third uncertainty is "intensity of exploration" and the remaining uncertainties are related to size and grade of the mineral deposit considered. The latter three uncertainties were illustrated by means of a case history study for large copper deposits in the Abitibi area of the Canadian Shield. Copper potential of this area was predicted on the basis of estimates of production and reserves that were available in 1968 totalling 3.12 Mt Cu. This prognostication could now be evaluated on the basis of copper production and reserves available in 2008 totalling 9.50 Mt Cu.

The method used consisted of relating a logarithmic measure of amount of copper for 10 km × 10 km cells to rock unit and geophysical variables and their cross-products by means of stepwise multiple regression. Felsic volcanics were most strongly represented in the probability index originally estimated in agreement with the fact the vast majority of large copper deposits in this region are volcanogenic massive sulphide deposits. A small minority of the deposits are magmatic Ni-Cu deposits primarily associated with mafic and ultramafic rocks. "Intensity of exploration" in Agterberg et al. (1972) was quantified by converting the probability index into a probability by defining a "control area" including the mining districts in 1968. In order to delineate target areas with undiscovered deposits it was assumed that all large copper deposits had been found within the "control area". This resulted in delineation of three regional copper anomalies. The earlier hindsight study performed 30 years ago on the basis of copper production and reserves available in 1977 (totalling 5.23 Mt Cu) showed seven new discoveries occurring either in the immediate vicinities of known deposits or on the prognostic regional copper anomalies. Post-1977 exploration resulted in six additional large copper deposits all relatively close to known deposits. Most of the increase from 3.12 to 9.50 Mt Cu was due to increased production and reserve estimates for known deposits. By 1977 the broad geographic distribution pattern of large copper deposits in the Abitibi area had stabilized. Much new copper was found at greater depths by deeper penetration of the Earth's crust.

Uncertainties associated with copper deposit size and grade were analyzed in the last part of this paper by comparing log-log plots for 1968 and 2008 copper cell size and ore weight versus rank and corresponding lognormal QQ-plots. Most of these plots show linear patterns indicating that both the Pareto and lognormal frequency distribution models can be used in a parametric approach. Straight lines fitted by least squares show that 1968 slopes provide good estimates of 2008 slopes. These slopes are determined by fractal dimension of the Pareto and logarithmic variance of the lognormal, respectively. On the other hand, the intercepts corresponding to these slopes showed significant increases related to the large increase in copper and ore production and reserves between 1968 and 2008. A new method based on comparing these increases, which was introduced at the end of the paper, showed that the Pareto is superior to the lognormal frequency distribution model.

It can be concluded that within the favorable environments in the Abitibi area, copper deposit size is controlled by a 2-parameter Pareto distribution. The fractal parameter (related to slope of best-fitting straight line on log size-log rank plot) is approximately independent of location but the other parameter (intercept of best-fitting straight line) is related to total volume of the block of favorable rock thoroughly explored for occurrences of large copper deposits. The practical significance of this result can be visualized as follows. Suppose both the Pareto and lognormal originally had been applied to the 1968 data set. Then, for the same total amount spent on exploration and ore reserve estimation during the next 40 years, the lognormal model would have underestimated 2008 copper ore production and reserves by approximately (3.030−1.831)/3.030 or 40 percent, whereas the Pareto model would have overestimated measured amounts of ore and copper by 8% and less than 1%, respectively.

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